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2001 J. Phys. A: Math. Gen. 34 8365

(http://iopscience.iop.org/0305-4470/34/40/312)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 8365–8382 PII: S0305-4470(01)24808-2

Deformations of the fermion realization of the $sp(4)$ algebra and its subalgebras

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Received 8 May 2001
Published 28 September 2001
Online at stacks.iop.org/JPhysA/34/8365

Abstract

With a view towards future applications in nuclear physics, the fermion realization of the compact symplectic $sp(4)$ algebra and its q-deformed versions are investigated. Three important reduction chains of the $sp(4)$ algebra are explored in both the classical and deformed cases. The deformed realizations are based on distinct deformations of the fermion creation and annihilation operators. For the primary reduction, the $su(2)$ substructure can be interpreted as either the spin, isospin or angular momentum algebra, whereas for the other two reductions $su(2)$ can be associated with pairing between fermions of the same type or pairing between two distinct fermion types. Each reduction provides for a complete classification of the basis states. The deformed induced $u(2)$ representations are reducible in the action spaces of $sp(4)$ and are decomposed into irreducible representations.

PACS numbers: 21.60.-n, 02.20.-a

1. Introduction

Symplectic algebras can be used to describe many-particle systems. The compact, $sp(2n)$, and noncompact versions, $sp(2n, R)$, of the algebra enter naturally when the number of particles or couplings between the particles changes in a pairwise fashion from one configuration to the next. In this paper we consider the simplest nontrivial case: the compact $sp(4)$ symplectic algebra, which is isomorphic to the Lie algebra of the five-dimensional rotation group $SO(5)$ [1–3]. Applications of $sp(4)$ are related to different interpretations of the quantum numbers of the fermions used to construct the generators of the $Sp(4)$ group.

Interest in symplectic groups is related to applications to nuclear structure [4–6]. In particular, $Sp(4)$ has been used to explore pairing correlations in nuclei [1,7,8]. The reduction chains to different realizations of the $u(2)$ subalgebra of $sp(4)$ yield a complete classification scheme for the basis states. It is rather easy to generalize this work to higher-rank algebras and therefore the algebraic techniques are illustrated by $sp(4)$ (e.g. [9]). A further interest in the symplectic algebras is related to their use in mapping methods from the fermion space to the space spanned by collective bosons and ideal fermions [10]. In these applications the primary purpose is to simplify the Hamiltonian of the initial problem.

In the last decade much effort, from a purely mathematical [11–14] as well as from the physical point of view [15], has been concentrated on various deformations of the classical Lie algebras. The general feature of these deformations is that at the limit of the deformation parameter $q \to 1$ the q-algebra reverts to the classical Lie algebra. More than one deformation can be realized for one and the same 'classical' algebra, which can be chosen in a convenient way in different physical applications. There are many similarities between the classical Lie algebras and their deformations, especially with respect to their representation. Deformed algebras introduce a new degree of freedom that can give a better explanation of nonlinear effects. Their study can lead to deeper understanding of the physical significance of the deformation.

In [16] a boson realization of the noncompact $sp(4, R)$ and two distinct deformations of it, as well as compact and noncompact subalgebras of each, were investigated and reductions of their action spaces obtained. As the fermion case has more direct application in nuclear theory than the boson construction, in this paper our aim is to investigate in detail the fermion realization of the $sp(4)$ algebra and its deformations. Using the methodology from [16], we begin with the well known realization of this algebra in terms of 'classical' fermion creation and annihilation operators and consider all the subalgebras which correspond to different ways of specifying labels of the basis states via the eigenvalues of the operators generating these subalgebras (section 2). Furthermore, we obtain the deformation of this $sp(4)$ algebra and its subalgebras by introducing a transformation function that deforms the classical fermions into q-deformed fermions. We also introduce another deformation in terms of the standard q-fermions and by following the same procedure we investigate the enveloping algebra of $sp(4)$ and the action of its generators on the deformed basis (section 3).

2. Fermion realization of the $sp(4)$ algebra

To establish the notation, recall some features of the fermion realization of the $sp(4)$ algebra [3,4,9], which is isomorphic to $so(5)$ [2], as normally used in the shell-model studies. The operator $c^{\dagger}_{m,\sigma}$ creates ($c_{m,\sigma}$ annihilates) a particle of type $\sigma = \pm 1$, in a state of total angular momentum $j = \frac{2k+1}{2}$, $k = 0, 1, 2, \ldots,$ with projection m along the z axis ($-j \leqslant m \leqslant j$). These operators satisfy Fermi anticommutation relations

$$\{c_{m',\sigma'}, c^{\dagger}_{m,\sigma}\} = \delta_{m',m}\delta_{\sigma',\sigma} \qquad \{c^{\dagger}_{m',\sigma'}, c^{\dagger}_{m,\sigma}\} = \{c_{m',\sigma'}, c_{m,\sigma}\} = 0 \tag{1}$$

and Hermitian conjugation is given by $(c^{\dagger}_{m,\sigma})^{*} = c_{m,\sigma}$.

For a given σ, the dimension of the fermion space is $2\Omega_j = 2j+1$. The fermion realization of $sp(4)$ is given in a standard way by means of the following operators [1,2,9]:

$$A_{\sigma,\sigma'} = \xi_{\sigma,\sigma'} \sum_{m=-j}^{j} (-1)^{j-m} c^{\dagger}_{m,\sigma} c^{\dagger}_{-m,\sigma'} = A_{\sigma',\sigma} = (B_{\sigma,\sigma'})^{\dagger} \tag{2}$$

$$B_{\sigma,\sigma'} = \xi_{\sigma,\sigma'} \sum_{m=-j}^{j} (-1)^{j-m} c_{-m,\sigma} c_{m,\sigma'} = B_{\sigma',\sigma} = (A_{\sigma,\sigma'})^{\dagger}. \tag{3}$$

These operators create (annihilate) a pair of fermions coupled to total angular momentum $J = 0$ [2] and thus constitute boson-like objects according to the spin-statistics theorem [17] when the operators

$$D_{\sigma,\sigma'} = \eta \sum_{m=-j}^{j} c^{\dagger}_{m,\sigma} c_{m,\sigma'} \tag{4}$$

preserve the number of fermions. Here the normalization constants are

$$\xi_{\sigma,\sigma'} = \frac{\eta}{\sqrt{(1+\delta_{\sigma,\sigma'})}} \qquad \eta = \frac{1}{\sqrt{2\Omega_j}}. \tag{5}$$

The number of the operators $A_{\sigma,\sigma'}$, $B_{\sigma,\sigma'}$ and $D_{\sigma,\sigma'}$ is ten ($A_{\sigma,\sigma'} = A_{\sigma',\sigma},\ B_{\sigma,\sigma'} = B_{\sigma',\sigma}$). Their commutation relations, obtained by means of (1), show that these operators generate a fermion realization of the $sp(4)$ algebra [1]. An additional index $m \neq 0$ of the creation and annihilation fermion operators is introduced in order to construct non-zero operators $A_{\sigma,\sigma}$ and $B_{\sigma,\sigma}$, but the index $\sigma = \pm 1$ defines the algebraic properties of the generators $A_{\sigma,\sigma'}$, $B_{\sigma,\sigma'}$ and $D_{\sigma,\sigma'}$.

Different interpretations of σ correspond to different physical meanings for the operators generating the ten-parametric $Sp(4)$ group and therefore different physical models. These can be used to describe various aspects of the nuclear interaction (different Hamiltonians) [9] like charge-independent pairing, two-level pairing (Lipkin model) or two-dimensional rotations and vibrations. The $sp(4)$ algebra is considered to be the dynamical symmetry algebra in these applications. Each of the limits is described by a reduction chain of the algebra, which serves to label the basis states by eigenvalues of the invariant operators of the subalgebras and gives the corresponding limiting forms of the model Hamiltonian.

2.1. Subalgebras of $sp(4)$

The investigation of the subalgebras of $sp(4)$ contained in its reduction chains is given below.

(1) By using the particle number preserving Weyl generators $D_{i,j}$ (4), a subalgebra $u(2)$ of $sp(4)$ is realized by the operators

$$\begin{aligned} &\tau_1 \equiv D_{+1,-1} \qquad && \tau_0 = \frac{N_1 - N_{-1}}{2} \\ &\tau_{-1} \equiv D_{-1,+1} \qquad && N = N_{+1} + N_{-1} \end{aligned} \tag{6}$$

where $N_{\pm 1} \equiv \frac{1}{\eta} D_{\pm 1,\pm 1}$ are the operators of the total number of fermions of each kind,

$$N_{\sigma} = \sum_{m=-j}^{j} c^{\dagger}_{m,\sigma} c_{m,\sigma} \qquad \sigma = \pm 1. \tag{7}$$

The action of these operators on the fermion creation and annihilation operators is given by

$$\begin{aligned} &N_{\sigma} c^{\dagger}_{m,\sigma'} = c^{\dagger}_{m,\sigma'}(N_{\sigma} + \delta_{\sigma,\sigma'}) \qquad N_{\sigma} c_{m,\sigma'} = c_{m,\sigma'}(N_{\sigma} - \delta_{\sigma,\sigma'}) \\ &\sigma, \sigma' = \pm 1 \end{aligned} \tag{8}$$

and the anticommutation relations (1) yield the equality

$$\sum_{m=-j}^{j} c_{m,\sigma} c^{\dagger}_{m,\sigma} = 2\Omega_j - N_{\sigma} \qquad \sigma = \pm 1. \tag{9}$$

The operators (6) satisfy the $u(2)$ commutation relations

$$[\tau_+, \tau_-] = 2\frac{\tau_0}{2\Omega_j} \qquad [\tau_0, \tau_{\pm}] = \pm\tau_{\pm} \qquad [N, \tau_{\pm}] = 0 \qquad [N, \tau_0] = 0 \tag{10}$$

where $\tau_0, \tau_\pm$ close on an algebra $su^\tau(2)$ that is isomorphic to $so(3)$. The operator N generates $U(1)$ and plays the role of the first-order invariant of $U^\tau(2) = SU^\tau(2) \otimes U(1)$. The second-order Casimir operator of $SU^\tau(2)$ is given by

$$\tau^2 = \frac{2\Omega_j}{2}(\tau_+\tau_- + \tau_-\tau_+) + \tau_0\tau_0 \tag{11}$$

and the second-order invariant of $U^\tau(2)$ [9] is simply

$$C_2 = N(N+1) - \tau^2. \tag{12}$$

The algebra $su^\tau(2) \backsim so(3)$ plays a very important role in all kinds of different physical applications since it is of the standard spin type, which can be interpreted as spin, isospin or angular momentum in the various models.

(2) Another unitary realization of $u(2)$, denoted by $u^0(2)$, is generated by τ_0 (6) and the operators

$$A^0_{+1} \equiv A_{1,-1} \qquad A^0_{-1} \equiv B_{1,-1} \qquad A^0_0 \equiv \frac{N}{2} - \Omega_j \tag{13}$$

with the following commutation relations:

$$\begin{aligned} &[A^0_{+1}, A^0_{-1}] = 2\frac{A^0_0}{2\Omega_j} \qquad &&[A^0_0, A^0_{\pm1}] = \pm A^0_{\pm1} \\ &[\tau_0, A^0_{\pm1}] = 0, &&[\tau_0, A^0_0] = 0. \end{aligned} \tag{14}$$

For this realization the operator τ_0 acts as a first-order invariant of $u^0(2)$, defining the reduction $u^0(2) = su^0(2) \oplus u^0(1)$. The second-order Casimir invariant of this subgroup is given as

$$C_2(SU^0(2)) = \frac{2\Omega_j}{2}(A^0_{+1}A^0_{-1} + A^0_{-1}A^0_{+1}) + A^0_0A^0_0. \tag{15}$$

The generators of this $SU^0(2)$ group are operators pairing particles of two different kinds.

(3) Next, we consider two mutually complementary $su(2)$ subalgebras of the algebra $sp(4)$, denoted by $su^+(2)$ and $su^-(2)$. These algebras are generated by the operators

$$A^\pm_{+1} \equiv A_{\pm1,\pm1} \qquad B^\pm_{-1} \equiv B_{\pm1,\pm1} \qquad D^\pm_0 \equiv \frac{N_{\pm1}}{2} - \frac{\Omega_j}{2} \tag{16}$$

with the following commutation relations:

$$[A^\pm_{+1}, B^\pm_{-1}] = 4\frac{D^\pm_0}{2\Omega_j} \qquad [D^\pm_0, A^\pm_{+1}] = A^\pm_{+1} \qquad [D^\pm_0, B^\pm_{-1}] = -B^\pm_{-1}. \tag{17}$$

It is simple to see that each of the generators of $SU^+(2)$ commutes with all of the $SU^-(2)$ generators. The second-order Casimir operators of the $SU^\pm(2)$ are

$$C_2(SU^\pm(2)) = \frac{\Omega_j}{2}(A^\pm_{+1}B^\pm_{-1} + B^\pm_{-1}A^\pm_{+1}) + D^\pm_0D^\pm_0. \tag{18}$$

In this case the addition of the operators $N_{\mp1}$, considered to be generators of the subgroups $U^\mp(1)$, extends $su^\pm(2)$ to $u^\pm(2) = su^\pm(2) \oplus u^\mp(1)$. $N_{\mp1}$ act as first-order Casimir operators of $U^\pm(2)$. The operators closing the two mutually complementary subalgebras describe pairs of particles of the same kind.

(4) The sum of the generators of the groups $SU^+(2)$ and $SU^-(2)$ gives rise to another unitary realization of the $su(2)$ subalgebra of $sp(4)$ denoted by $\widetilde{su(2)}$,

$$\tilde{A}_{+1} \equiv A^+_{+1} + A^-_{+1} \qquad \tilde{B}_{-1} \equiv B^+_{-1} + B^-_{-1} \qquad \tilde{D}_0 \equiv \frac{N_1}{2} + \frac{N_{-1}}{2} - \Omega_j \tag{19}$$

with the commutation relations

$$[\tilde{A}_{+1}, \tilde{B}_{-1}] = 4\frac{\tilde{D}_0}{2\Omega_j} \qquad [\tilde{D}_0, \tilde{A}_{+1}] = \tilde{A}_{+1} \qquad [\tilde{D}_0, \tilde{B}_{-1}] = -\tilde{B}_{-1} \tag{20}$$

and the second-order Casimir invariant

$$C_2(\widetilde{SU(2)}) = \frac{\Omega_j}{2}(\tilde{A}_{+1}\tilde{B}_{-1} + \tilde{B}_{-1}\tilde{A}_{+1}) + \tilde{D}_0\tilde{D}_0. \tag{21}$$

2.2. *Action space of the fermion realization of sp(4)*

In general, the classical fermion operators act in a finite space $\mathcal{E}_j$ for a particular j-level. The finite representation is due to the Pauli principle, $c^\dagger_{m,\sigma}c^\dagger_{m,\sigma}|0\rangle = 0$, that allows no more than $2\Omega_j$ identical fermions in a single j-shell. In $\mathcal{E}_j$ the vacuum $|0\rangle$ is defined by $c_{m,\sigma}\ |0\rangle = 0$ and the scalar product is chosen so that $\langle 0|0\rangle = 1$.

The states that span the $\mathcal{E}_j$ spaces consist of different numbers of fermion creation operators acting on the vacuum state. These satisfy the Pauli principle through their anti-commutation relations (1). They form an orthonormal basis in each space and are the common eigenvectors of the fermion number operators N_1, N_{-1} ($N_\sigma = N^*_\sigma, \sigma = \pm 1$) and $N = N_1 + N_{-1}$. In this way, they span two subspaces $\mathcal{E}^\pm_j$ labelled by the eigenvalue of the invariant operator $P = (-1)^N$ of $Sp(4)$. Here we are interested in the even space $\mathcal{E}^+_j$, containing states of coupled fermions, in order to apply the theory to phenomena like pairing correlations in nuclei.

If we introduce

$$A^\dagger_{\frac{1}{2}(\sigma+\sigma')} \equiv A_{\sigma,\sigma'} \qquad B_{-\frac{1}{2}(\sigma+\sigma')} \equiv B_{\sigma,\sigma'} \qquad \sigma, \sigma' = \pm 1 \tag{22}$$

for operators creating (2) and annihilating (3) a pair of particles, it is easy to check that they are components of two conjugated vectors $\{A^\dagger_k\}_{k=0,\pm1}$ and $\{B_{-k}\}_{k=0,\pm1}$, $k = \frac{1}{2}(\sigma+\sigma') = 0, \pm 1$ with respect to the subgroup $SU^\tau(2)$ ((6), (10)):

$$\begin{aligned} &[\tau_0, A^\dagger_k] = kA^\dagger_k \qquad [\tau_l, A^\dagger_k] = \frac{1}{\sqrt{\Omega_j}} A^\dagger_{l+k} \qquad l = \pm 1 \\ &[\tau_0, B_k] = kB_k \qquad [\tau_l, B_k] = -\frac{1}{\sqrt{\Omega_j}} B_{l+k} \qquad l = \pm 1. \end{aligned} \tag{23}$$

In the models where $\boldsymbol{\tau}$ is interpreted as the isospin operator, $A^\dagger_{0,\pm1}\{B_{0,\mp1}\}$ create (destroy) a pair of fermions coupled to a total isospin $\tau = 1$.

Thus, a linearly independent set of vectors that span the $\mathcal{E}^+_j$ space can be expressed in terms of the 'boson creation operators' acting on the vacuum state,

$$|\Omega_j; n_1, n_0, n_{-1}) = (A^\dagger_1)^{n_1}(A^\dagger_0)^{n_0}(A^\dagger_{-1})^{n_{-1}}|0\rangle. \tag{24}$$

The basis is obtained by orthonormalization of (24). The operators $A^\dagger_{0,\pm1}$ commute among themselves and therefore form a symmetric representation. The eigenvalue of the second-order Casimir operator Ω_j labels each representation of $Sp(4)$,

$$\begin{aligned} C_2(Sp(4)) = &\{\tau_+, \tau_-\} + \{A^0_{+1}, A^0_{-1}\} + \{A^+_{+1}, B^+_{-1}\} + \{A^-_{+1}, B^-_{-1}\} \\ &+ \frac{1}{\Omega_j}\left(\tau_0\tau_0 + \frac{(N - 2\Omega_j)^2}{4}\right) \end{aligned} \tag{25}$$

$$C_2(Sp(4))|\Omega_j; n_1, n_0, n_{-1}) = (\Omega_j + 3)|\Omega_j; n_1, n_0, n_{-1}). \tag{26}$$

Table 1. Basis sets for the $Sp(4)$ representation, $\Omega_{312} = 2$.

n	$i = 2$	$i = 1$	$i = 0$	$i = -1$	$i = -2$
0			$\|0, 0, 0)$		
2		$\|1, 0, 0)$	$\|0, 1, 0)$	$\|0, 0, 1)$	
4	$\|2, 0, 0)$	$\|1, 1, 0)$	$\|1, 0, 1)$ $\|0, 2, 0)$	$\|0, 1, 1)$	$\|0, 0, 2)$
6		$\|2, 0, 1)$ $\|1, 2, 0)$	$\|1, 1, 1)$ $\|0, 3, 0)$	$\|1, 0, 2)$ $\|0, 2, 1)$	
8			$\|2, 0, 2)$ $\|1, 2, 1)$ $\|0, 4, 0)$		

The group $Sp(4)$ is of rank two and thus there exist two invariant operators that commute with all the generators of the group [1]. The other invariant operator is of fourth order and it is linearly dependent on the Casimir operator for this group. Usually representations of $Sp(4)$ are labelled by the largest eigenvalue of the number operator N and the reduced isospin of the uncoupled fermions in the corresponding state [2, 18]. In each representation of $Sp(4)$ in the vector space spanned over (24), the maximum number of particles is $4\Omega_j$ and the respective state consists of no uncoupled fermions (reduced isospin zero). It follows that only one quantum number is needed, Ω_j. Within a representation, Ω_j is dropped from the labelling of the states. Another consequence of the symmetric representation is that the vector space consists of states of a system with total angular momentum $J = 0^+$.

Each representation labelled by Ω_j is finite, because of the fermion structure of the operators $A^\dagger_{0,\pm1}$: $(A^\dagger_{\pm1})^{\Omega_j+1}|0\rangle = 0$ or $(A^\dagger_{\pm1})^{\Omega_j}(A^\dagger_0)|0\rangle = 0$. Another consequence of the fermion realization is that some of the vectors (24) of the finite space $\mathcal{E}^+_j$ are linearly dependent, for example $(A^\dagger_1)^{\Omega_j}(A^\dagger_{-1})^{\Omega_j}|0\rangle \sim (A^\dagger_0)^{2\Omega_j}|0\rangle$.

The states (24) are the common eigenvectors of the fermion number operators N_1, N_{-1} ($N_\sigma = N^*_\sigma, \sigma = \pm1$):

$$\begin{aligned} N_1|n_1, n_0, n_{-1}) &= (2n_1 + n_0)|n_1, n_0, n_{-1}) \\ N_{-1}|n_1, n_0, n_{-1}) &= (2n_{-1} + n_0)|n_1, n_0, n_{-1}) \end{aligned} \tag{27}$$

or of the operators $N = N_1 + N_{-1}$ and $\tau_0 = \frac{1}{2}(N_1 - N_{-1})$, which are both diagonal in the basis (24)

$$N|n_1, n_0, n_{-1}) = n|n_1, n_0, n_{-1}) \qquad n = 2(n_1 + n_{-1} + n_0) \tag{28}$$

$$\tau_0|n_1, n_0, n_{-1}) = i|n_1, n_0, n_{-1}) \qquad i = n_1 - n_{-1}. \tag{29}$$

Their eigenvalues can be used to classify the basis within a representation Ω_j. The basis states labelled by $|n_1, n_0, n_{-1})$ for $\Omega_{3/2} = 2$ are shown in table 1, where n enumerates the rows and i the columns.

The basis vectors are degenerate in the sense that more than one of the common eigenstates of the operators N and τ_0 have one and the same eigenvalues $\{n, i\}$ and thus belong to one and the same cell of table 1. They can be distinguished as eigenstates of the Casimir operators of the limiting cases:

(1) The basis states can be labelled by the eigenvalues of the invariant operator of each subgroup in the reduction $Sp(4) \supset U(2) \supset SU^\tau(2) \otimes U_N(1)$. As a first-order invariant of $U(2)$, the operator N decomposes the spaces $\mathcal{E}^+_j$ into a direct sum of eigensubspaces, defined by the condition that n is fixed (28),

$$n = 0, 2, 4, \ldots, 4\Omega_j. \tag{30}$$

Table 2. Basis sets for the $SU^{\tau}(2)$ representation, $\Omega_{312} = 2$.

$n_{(\tau)}$	$i = 2$	$i = 1$	$i = 0$	$i = -1$	$i = -2$
0 $\overset{\tau=0}{\rightarrow}$			$\|0, 0, 0)$		
2 $\overset{\tau=1}{\rightarrow}$		$\|1, 0, 0)$	$\|0, 1, 0)$	$\|0, 0, 1)$	
$\overset{\tau=2}{\rightarrow}$	$\|2, 0, 0)$	$\sqrt{2}\|1, 1, 0)$	$\frac{\|0,2,0)}{\sqrt{3/2}} + \frac{\|1,0,1)}{\sqrt{3/2}}$	$\sqrt{2}\|0, 1, 1)$	$\|0, 0, 2)$
4 $\overset{\tau=0}{\rightarrow}$			$\frac{\|0,2,0)}{\sqrt{15/2}} - \frac{2\|1,0,1)}{\sqrt{15/2}}$		
6 $\overset{\tau=1}{\rightarrow}$		$\|2, 0, 1) \equiv$ $-2\|1, 2, 0)$	$2\|1, 1, 1) \equiv$ $\frac{-2}{3}\|0, 3, 0)$	$\|1, 0, 2) \equiv$ $-2\|0, 2, 1)$	
8 $\overset{\tau=0}{\rightarrow}$			$\frac{2}{3}\|0, 4, 0) \equiv$ $-2\|1, 2, 1) \equiv$ $\|2, 0, 2)$		

So an irreducible unitary representation (IUR) of $U(2)$ is realized in each row of table 1. The $SU^{\tau}(2)$ subgroup provides the other two quantum numbers as a standard label of the basis vectors. First, it is the eigenvalue of the Casimir operator of second rank in $\boldsymbol{\tau}$,

$$\boldsymbol{\tau}^2|n, \tau, i\rangle = \tau(\tau + 1)|n, \tau, i\rangle \tag{31}$$

where $\tau = \frac{\tilde{n}}{2}, \frac{\tilde{n}}{2} - 2, \ldots, 1$ (odd) or 0 (even), $\tilde{n} = \min\{n, 4\Omega_j - n\}$, and second it is the eigenvalue of τ_0 (29), where $i = n_1 - n_{-1} = -\tau, -\tau + 1, \ldots, \tau$. As an example, the orthonormalized basis $|n, \tau, i\rangle$ given in terms of the states $|n_1, n_0, n_{-1})$ is shown in table 2 for $\Omega_{3/2} = 2$.

In general, the state with the maximum number of particles always has a total isospin zero, $\tau = 0$, and all the possible states expressed in the basis $|n_1, n_0, n_{-1})$ are equivalent within a normalization factor. The raising (lowering) generators $\tau_{\pm 1}$ acting $(2\tau + 1)$ times on the lowest $|n, \tau, -\tau\rangle$ (highest $|n, \tau, \tau\rangle$) weight state give all the basis states of the respective τ-representation according to the result

$$\tau_{\pm 1}|n, \tau, i\rangle = \sqrt{\frac{(\tau \mp i)(\tau \pm i + 1)}{2\Omega_j}}|n, \tau, i \pm 1\rangle. \tag{32}$$

(2) The reduction chain $Sp(4) \supset U^0(2) \supset SU^0(2) \otimes U_{\tau_0}(1)$ introduces another labelling scheme for the basis states, namely, $|i, 2(n_1 + n_{-1})_{\max}, n)$. The quantum numbers that specify the states are the eigenvalue i of τ_0, $i = -\Omega_j, -\Omega_j + 1, \ldots, \Omega_j$, the seniority quantum number $\nu = 2(n_1 + n_{-1})_{\max} = 2|i|$, and the eigenvalue of the operator $A_0^0 = -(\Omega_j - |i|), -(\Omega_j - |i|) + 1, \ldots, (\Omega_j - |i|)$ (13). The first invariant of $U^0(2)$ decomposes the spaces $\mathcal{E}_j^+$ into a direct sum of eigensubspaces of the operator τ_0 at each of its fixed values (29). These subspaces are represented by the columns of table 1. The operator A_0^0 (13) does not differ essentially from the first invariant operator N of $U_N(1)$ and it further reduces the columns of table 1 to the cells. The seniority quantum number differs between two states of one and the same i and n, but different coupling schemes, and it is introduced by the eigenvalues of the second Casimir operator for this subgroup:

$$\begin{aligned} C_2(SU^0(2))|n_1, n_0, n_{-1}) = {} & \frac{2\Omega_j - 2(n_1 + n_{-1})_{\max}}{2} \\ & \times \left(\frac{2\Omega_j - 2(n_1 + n_{-1})_{\max}}{2} + 1\right)|n_1, n_0, n_{-1}). \end{aligned} \tag{33}$$

This is a scheme for coupling particles of the two different kinds $\{\sigma = 1, \sigma' = -1\}$ with $n_1 = 0$, or $n_{-1} = 0$, or both $n_1 = n_{-1} = 0$. These states are the last ones in each of the cells in the table 1. The additional quantum number, $\nu = 2(n_1 + n_{-1})_{\max}$, is the maximum number of the remaining pairs coupled as $\{\sigma = 1, \sigma' = 1\}$ or $\{\sigma = -1, \sigma' = -1\}$. In that limit, the Casimir operator can be expressed in terms of the eigenvalue of the first-order invariant of $U^0(2)$,

$$C_2(SU^0(2))|n_1, n_0, n_{-1}) = (\Omega_j - |i|)(\Omega_j - |i| + 1)|n_1, n_0, n_{-1}). \tag{34}$$

The raising and lowering generators of the subgroup $SU^0(2)$ act along the columns in the following way:

$$\begin{aligned} A^0_{+1}|n_1, n_0, n_{-1}) &= |n_1, n_0 + 1, n_{-1}) \\ A^0_{-1}|n_1, n_0, n_{-1}) &= n_0 \left(1 - \frac{2(n_{-1} + n_1) + n_0 - 1}{2\Omega_j}\right)|n_1, n_0 - 1, n_{-1}). \end{aligned} \tag{35}$$

In each column i, A^0_{+1} starts from the lowest-weight state $|n_1, n_0, 0)$ or $|0, n_0, n_{-1})$, $n_0 = 0, n_{\pm 1} = 0, 1, \ldots, |i|$ and gives all the basis states within a τ_0-representation with $n_0 = 1, 2, \ldots, 2(\Omega_j - |i|)$. Similarly, A^0_{-1} gives the basis states of the representation of the subgroup under consideration, starting with the highest-weight state $n_0 = 2(\Omega_j - |i|)$, for each i.

The normalized basis states,

$$|n_1, n_0, n_{-1}\rangle = \frac{1}{\mathcal{N}_0(n_1, n_0, n_{-1})}|n_1, n_0, n_{-1}) \tag{36}$$

can be derived from (35). For the three types of state in this reduction, the normalization coefficients are given by

$$\begin{aligned} &\mathcal{P}^2_0(n_1, n_0, n_{-1}) = n_0! \prod_{k=0}^{n_0-1}\left(1 - \frac{2(n_{-1} + n_1) + k}{2\Omega_j}\right) \\ &\mathcal{N}_0(0, n_0, 0) = \mathcal{P}_0(0, n_0, 0) \\ &\mathcal{N}_0(n_1, n_0, 0) = \mathcal{P}_0(n_1, n_0, 0)\mathcal{N}(n_1) \\ &\mathcal{N}_0(0, n_0, n_{-1}) = \mathcal{P}_0(0, n_0, n_{-1})\mathcal{N}(n_{-1}) \end{aligned} \tag{37}$$

where the lowest-weight state ($n_0 = 0$ and $n_{\mp 1} = 0$) in each representation can be normalized recursively,

$$\mathcal{N}^2(n_{\pm 1}) = n_{\pm 1}! \prod_{l=0}^{n_{\pm 1}-1}\left(1 - \frac{l}{\Omega_j}\right). \tag{38}$$

(3) The other reduction is described again by the invariants of the subgroups in the reduction chain: $Sp(4) \supset U^{\pm}(2) \supset SU^{\pm}(2) \otimes U(1)_{N_{\mp}}$. Here the labelling is $|n_{\mp 1}, n_0 = n_{0\max}, n_{\pm 1})$. First the spaces $\mathcal{E}^+_j$ are decomposed by means of the first-order invariants $N_{\mp}$ of the respective subalgebras to the subspaces defined by the conditions $(2n_{\mp 1} + n_0) = 0, 1, \ldots, 2\Omega_j$ and represented by the left (right) diagonals in table 1. The action of the Casimir operator on the states

$$C_2(SU^{\pm}(2))|n_1, n_0, n_{-1}) = \frac{\Omega_j - n_{0\max}}{2}\left(\frac{\Omega_j - n_{0\max}}{2} + 1\right)|n_1, n_0, n_{-1}) \tag{39}$$

provides the $su(2)$ quantum number $d = (\Omega - n_{0\max})/2$. The seniority quantum number $n_{0\max}$ is the maximum number of remaining pairs that can be formed by coupling particles of different types, here $n_{0\max} = \{0 \text{ or } 1\}$. The basis states are of the form $(A_1{}^{\dagger})^{n_1}(A_{-1}{}^{\dagger})^{n_{-1}}|0\rangle$ and $(A_1{}^{\dagger})^{n_1}A_0{}^{\dagger}(A_{-1}{}^{\dagger})^{n_{-1}}|0\rangle$ and they are placed first in

each cell in table 1. Furthermore, the operators $D_0^{\pm}$ (16), which are equivalent within constants to the operators $N_{\pm}$, give the respective projection of d: $D_0^{\pm}|n_1, n_0, n_{-1}) = \frac{1}{2}(2n_{\pm} + n_{0\,\max} - \Omega_j)|n_1, n_0, n_{-1}) = d_0^{\pm}|n_1, n_0, n_{-1})$. The diagonals are decomposed to the cells belonging to them and defined by the conditions $d_0^{\pm} = -d, -d+1, \ldots, d$.
The raising and lowering generators of $SU^{\pm}(2)$ act along the left/right diagonals:

$$\begin{aligned} A_{+1}^{\pm}|n_1, n_0, n_{-1}) &= |n_{\pm 1} + 1, n_0, n_{\mp 1}) \\ B_{-1}^{\pm}|n_1, n_0, n_{-1}) &= n_{\pm 1}\left(1 - \frac{n_{\pm 1} + n_0 - 1}{\Omega_j}\right)|n_{\pm_1} - 1, n_0, n_{\mp 1}). \end{aligned} \tag{40}$$

Starting from the respective lowest- or highest-weight states, they generate all the states belonging to IURs of the $U^{\pm}(2)$ subgroups of $Sp(4)$.
The normalized basis states,

$$|n_1, n_0, n_{-1}\rangle = \frac{1}{\mathcal{N}_{\pm}(n_1, n_0, n_{-1})}|n_1, n_0, n_{-1}) \tag{41}$$

can be derived from (40). For the two types of state $n_0 = \{0 \text{ or } 1\}$ in this reduction, the normalization coefficients are given by

$$\mathcal{N}_{\pm}^2(n_1, n_0, n_{-1}) = n_1!n_{-1}!\prod_{l=0}^{n_1-1}\left(1 - \frac{n_0 + l}{\Omega_j}\right)\prod_{l=0}^{n_{-1}-1}\left(1 - \frac{n_0 + l}{\Omega_j}\right) \tag{42}$$

where the results (42) are consistent with (38) for $n_0 = 0$ and with the lowest-weight state in each representation ($n_{\pm 1} = 0$) normalized recursively:

$$\mathcal{N}^2(n_{\mp 1}, n_0) = n_0!\prod_{l=0}^{n_0-1}\left(1 - \frac{2n_{\mp 1} + l}{2\Omega_j}\right)n_{\mp 1}!\prod_{l=0}^{n_{\mp 1}-1}\left(1 - \frac{l}{\Omega_j}\right). \tag{43}$$

3. q-deformations of the fermion realization of $sp(4)$

Consider q-deformed creation and annihilation operators $\alpha_{m,\sigma}^{\dagger}$ and $\alpha_{m,\sigma}$, $m = -j, -j+1, \ldots, j$, $\sigma = \pm 1$, for a particle of type σ in a state of total angular momentum j, with projection m on the z axis. The Hermitian conjugation relation is defined as $(\alpha_{m,\sigma}^{\dagger})^* = \alpha_{m,\sigma}$.

3.1. q-deformed transformation of the fermion operators

There is a general class of functions which transform the classical operators into deformed ones [19, 20]. We use the transformation

$$\alpha_{m,\sigma} = \theta^{\frac{N_\sigma}{2}} c_{m,\sigma} \qquad \alpha_{m,\sigma}^{\dagger} = c_{m,\sigma}^{\dagger}\bar{\theta}^{\frac{N_\sigma}{2}} \tag{44}$$

where θ is a complex number with amplitude $|\theta| = q$, q a real number, and $N_\sigma = \sum_m N_{m,\sigma}$ are the classical number operators. The transformation of (1) leads to the anticommutation relations for the q-deformed fermion operators,

$$\alpha_{m,\sigma}\alpha_{m,\sigma}^{\dagger} + q\alpha_{m,\sigma}^{\dagger}\alpha_{m,\sigma} = q^{N_\sigma} \tag{45}$$

and the identities

$$\begin{aligned} \sum_m \alpha_{m,\sigma}^{\dagger}\alpha_{m,\sigma} &= N_\sigma q^{N_\sigma - 1} \\ \sum_m \alpha_{m,\sigma}\alpha_{m,\sigma}^{\dagger} &= (2\Omega_j - N_\sigma)q^{N_\sigma}. \end{aligned} \tag{46}$$

The raising and lowering generators of the respective deformed $Sp(4)$ group are given as in the classical case (2)–(4) but in terms of the q-deformed fermion operators:

$$F_{\sigma,\sigma'} = \xi_{\sigma,\sigma'} \sum_{m=-j}^{j} (-1)^{j-m} \alpha^{\dagger}_{m,\sigma} \alpha^{\dagger}_{-m,\sigma'} = F_{\sigma',\sigma} = (G_{\sigma,\sigma'})^{\dagger} \tag{47}$$

$$G_{\sigma,\sigma'} = \xi_{\sigma,\sigma'} \sum_{m=-j}^{j} (-1)^{j-m} \alpha_{-m,\sigma} \alpha_{m,\sigma'} \tag{48}$$

and

$$E_{1,-1} = \eta \sum_{m=-j}^{j} \alpha^{\dagger}_{m,1} \alpha_{m,-1} \qquad E_{-1,1} = \eta \sum_{m=-j}^{j} \alpha^{\dagger}_{m,-1} \alpha_{m,1} \tag{49}$$

where the constants are defined in (5). The operator $F_{\sigma\sigma}$ $(G_{\sigma\sigma})$ creates (destroys) a q-deformed pair of particles of the same kind.

The remaining two Cartan generators N_σ, $\sigma = \pm 1$, used in the deformed commutation relations (45), are not deformed. The transformation (44) yields the following relations between the deformed (47)–(49) and the classical operators (2)–(4):

$$F_{\sigma,\sigma} = A_{\sigma,\sigma} \bar{\theta}^{N_\sigma + \frac{1}{2}} \qquad F_{1,-1} = A_{1,-1} \bar{\theta}^{\frac{N}{2}} \tag{50}$$

$$G_{\sigma,\sigma} = \theta^{N_\sigma + \frac{1}{2}} B_{\sigma,\sigma} \qquad G_{1,-1} = \theta^{\frac{N}{2}} B_{1,-1} \tag{51}$$

and

$$E_{1,-1} = D_{1,-1} \theta^{\frac{N_- - 1}{2}} \bar{\theta}^{\frac{N_+}{2}} \qquad E_{-1,1} = D_{-1,1} \theta^{\frac{N_+ - 1}{2}} \bar{\theta}^{\frac{N_-}{2}}. \tag{52}$$

Since there is a smooth transformation that depends on the Cartan generators of $sp(4)$ only and maps the q-deformed operators

$$\begin{array}{ll} F^{\dagger}_{k} = F^{\dagger}_{\frac{1}{2}(\sigma+\sigma')} \equiv F_{\sigma,\sigma'} & (G_{-k} = G_{-\frac{1}{2}(\sigma+\sigma')} \equiv G_{\sigma,\sigma'}) \\ \sigma, \sigma' = \pm 1 & k = \frac{1}{2}(\sigma + \sigma') = 0, \pm 1 \end{array} \tag{53}$$

to the classical vectors $A^{\dagger}_{0,\pm 1}$ $(B_{0,\mp 1})$, the q-deformed states are equivalent within a phase to the classical ones (24). All the relations revert to the classical formulae in the limit $q \to 1$. The important reduction of $sp_q(4)$ algebra to compact $u_q(2)$ subalgebra can be used again to obtain classification schemes for the basis states.

(1) The subalgebra $u_q(2)$ of $sp_q(4)$ is closed by the number preserving Weyl generators (49) and N_σ, $\sigma = \pm 1$, defined as

$$\begin{array}{ll} T_+ \equiv E_{1,-1} & T_0 \equiv \tau_0 = \dfrac{N_1 - N_{-1}}{2} \\ T_- \equiv E_{-1,1} & N = N_1 + N_{-1}. \end{array} \tag{54}$$

The generators T_0, $T_{\pm 1}$ and N satisfy the commutation relations

$$\begin{array}{ll} [T_1, T_{-1}] = \dfrac{T_0}{\Omega_j} q^{N-1} & [T_0, T_{\pm 1}] = \pm T_{\pm 1} \\ [N, T_{\pm 1}] = 0 & [N, T_0] = 0 \end{array} \tag{55}$$

and the second invariant of $u_q(2)$ is

$$C_2 = N(N+1) - \boldsymbol{T}^2. \tag{56}$$

The q-deformed operator $\boldsymbol{T}^2$ is defined by

$$\begin{aligned} \boldsymbol{T}^2 &= \frac{2\Omega_j}{2}(T_1 T_{-1} + T_{-1} T_1) + T_0 T_0 q^{N-1} \\ &= 2\Omega_j T_{-1} T_1 + T_0 (T_0 + 1) q^{N-1} \end{aligned} \tag{57}$$

and is related to the classical Casimir operator of $SU^{\tau}(2)$ (11) by

$$\boldsymbol{T}^2 = \boldsymbol{\tau}^2 q^{N-1}. \tag{58}$$

Thus, the eigenvalues of the Casimir operator are deformed by a phase factor q^{n-1} and the eigenvectors are the classical basis states, $|n, \tau, i\rangle$.

(2) The other subgroup $U_q^0(2)$ is generated by the operators

$$K_{+1}^0 \equiv F_{1,-1} \qquad K_{-1}^0 \equiv G_{1,-1} \qquad K_0^0 \equiv \frac{N}{2} - \Omega_j \tag{59}$$

and T_0 (54), which is the first-order invariant. The generators of $SU_q^0(2)$ commute in the following way:

$$[K_{+1}^0, K_{-1}^0]_{-2} = \frac{K_0^0}{\Omega_j} q^{N-2} \qquad [K_0^0, K_{\pm1}^0] = \pm K_{\pm1}^0 \tag{60}$$

where the q-commutator is defined as

$$[A, B]_k = AB - q^k BA. \tag{61}$$

The second-order Casimir invariant of $su_q^0(2)$ is given by

$$\begin{aligned} C_2(SU^0(2)) &= \frac{2\Omega_j}{2}(q^2 K_{+1}^0 K_{-1}^0 + K_{-1}^0 K_{+1}^0) q^{-N} + (K_0^0)^2 \\ &= 2\Omega_j K_{-1}^0 K_{+1}^0 q^{-N} + K_0^0(K_0^0 + 1). \end{aligned} \tag{62}$$

(3) The two mutually complementary subalgebras $su_q^+(2)$ and $su_q^-(2)$ of the algebra $sp_q(4)$ are given by the q-deformed operators

$$F_{+1}^{\pm} = F_{\pm1,\pm1} \qquad G_{-1}^{\pm} = G_{\pm1,\pm1} \tag{63}$$

and the nondeformed Cartan operators

$$E_0^{\pm} = \frac{N_{\pm1}}{2} - \frac{\Omega_j}{2}. \tag{64}$$

According to the reduction chain $Sp_q(4) \supset SU_q^{\pm}(2) \otimes U(1)_{N_{\mp}}$, $N_{\mp}$ commute with the operators $F_{+1}^{\pm}, G_{-1}^{\pm}, E_0^{\pm}$, which close the $su_q^{\pm}(2)$ algebra:

$$[F_{+1}^{\pm}, G_{-1}^{\pm}]_{-4} = \frac{2E_0^{\pm}}{\Omega_j}\, q^{2N_{\pm1}-3} \tag{65}$$

$$[N_{\pm1}, F_{+1}^{\pm}] = 2F_{+1}^{\pm} \qquad [N_{\pm1}\, , G_{-1}^{\pm}] = -2G_{-1}^{\pm}. \tag{66}$$

The corresponding Casimir invariant is

$$\begin{aligned} C_2(SU^{\pm}(2)) &= \frac{\Omega_j}{2}(q^4 F_{+1}^{\pm} G_{-1}^{\pm} + G_{-1}^{\pm} F_{+1}^{\pm}) q^{-2N_{\pm1}-1} + (E_0^{\pm})^2 \\ &= \Omega_j G_{-1}^{\pm} F_{+1}^{\pm} q^{-2N_{\pm1}-1} + E_0^{\pm}(E_0^{\pm} + 1). \end{aligned} \tag{67}$$

A similar q-deformation is based on the transformation $\alpha_{m,\sigma} = \theta^{-\frac{N_\sigma}{2}} c_{m,\sigma}$, which yields the same relations and identities as above, but with the exchange $q \to q^{-1}$. When θ is real and positive the deformation parameter is $\theta \equiv q$.

3.2. q-deformation of the anticommutation relations of the fermion operators

Consider another set of q-deformed Hermitian conjugate operators $\alpha^\dagger_{m,\sigma}$ and $\alpha_{m,\sigma}$, $(\alpha^\dagger_{m,\sigma})^* = \alpha_{m,\sigma}$, $m = -j, -j+1, \ldots, j$, $\sigma = \pm 1$. Let the q-deformed anticommutation relation hold for every σ and m in the form [14, 21]:

$$\alpha_{m,\sigma}\alpha^\dagger_{m,\sigma} + q^{\pm 1}\alpha^\dagger_{m,\sigma}\alpha_{m,\sigma} = q^{\pm N_{m,\sigma}} \tag{68}$$

where $N_{m,\sigma} = c^\dagger_{m,\sigma}c_{m,\sigma}$ and $N_\sigma = \sum_{m=-j}^{j} N_{m,\sigma}$ are the classical number operators (7). Their action on the deformed fermion operators is defined as in the classical case (8):

$$[N_\sigma, \alpha^\dagger_{m,\sigma'}] = \delta_{\sigma,\sigma'}\alpha^\dagger_{m,\sigma'} \qquad [N_\sigma, \alpha_{m,\sigma'}] = -\delta_{\sigma,\sigma'}\alpha_{m,\sigma'} \qquad \sigma, \sigma' = \pm 1. \tag{69}$$

In the previous section we showed that if the transformation function (44) is used, the anticommutation relations of the deformed fermion operators (45) depend not only on a single term $N_{m,\sigma}$ as in (68) but rather on the total sum N_σ. The same dependence, along with the requirement that the deformation is performed only on the σ index, defines

$$\alpha_{m,\sigma}\alpha^\dagger_{m,\sigma} + q^{\pm 1}\alpha^\dagger_{m,\sigma}\alpha_{m,\sigma} = q^{\pm\frac{N_\sigma}{2\Omega_j}}. \tag{70}$$

Using both anticommutation relations, it follows that $\alpha^\dagger_{m,\sigma}\alpha_{m,\sigma} = [\frac{N_\sigma}{2\Omega_j}]$, where $[X] = \frac{q^X - q^{-X}}{q - q^{-1}}$, which leads to

$$\sum_m \alpha^\dagger_{m,\sigma}\alpha_{m,\sigma} = 2\Omega_j\left[\frac{N_\sigma}{2\Omega_j}\right] \tag{71}$$

and

$$\sum_m \alpha_{m,\sigma}\alpha^\dagger_{m,\sigma} = 2\Omega_j\left[1 - \frac{N_\sigma}{2\Omega_j}\right]. \tag{72}$$

In the limit $q \to 1$, presuming $\alpha^\pm_{m,\sigma} \to c^\pm_{m,\sigma}$ as well, (71), (72) revert to the classical formulae for N_σ (7), (9). This justifies the introduction of the weight coefficient $\omega \equiv 1/(2\Omega_j)$ in (70). The remaining anticommutation relations for the q-deformed operators can be chosen from among various possibilities [22, 23]:

$$\begin{aligned} &\{\alpha_{m,\sigma}, \alpha^\dagger_{m',\sigma}\}_{q^{\pm 1}} = q^{\pm\frac{N_\sigma}{2\Omega_j}}\delta_{m,m'} && \{\alpha_{m,\sigma}, \alpha^\dagger_{m',\sigma'}\} = 0 \qquad \sigma \neq \sigma' \\ &\{\alpha^\dagger_{m,\sigma}, \alpha^\dagger_{m',\sigma'}\} = 0 && \{\alpha_{m,\sigma}, \alpha_{m',\sigma'}\} = 0 \end{aligned} \tag{73}$$

where the q-anticommutator is given by $\{A, B\}_k = AB + q^k BA$.

The set of generators for this realization of the deformed $sp_q(4)$ algebra is defined as in (47)–(49), but in terms of the q-deformed creation and annihilation operators $\alpha^\dagger_{m,\sigma}(\alpha_{m,\sigma})$, fulfilling anticommutation relations (73). The Cartan generators $N_{\pm 1}$ remain the classical number operators. These ten operators generate the q-deformed $Sp_q(4)$ group and its subgroup structure is investigated in analogy with the classical case.

(1) The subgroup $U_q(2)$ of $Sp_q(4)$ is generated by the number preserving Weyl operators (49) and N_σ, $\sigma = \pm 1$, as well as by the equivalent set of the operators $T_{0,\pm 1}$ and N (54). These operators satisfy the commutation relations

$$\begin{aligned} &[T_+, T_-] = \left[2\frac{T_0}{2\Omega_j}\right] && [T_0, T_\pm] = \pm T_\pm \\ &[N, T_\pm] = 0 && [N, T_0] = 0. \end{aligned} \tag{74}$$

The operators $T_{0,\pm 1}$ close an algebra $su_q(2) \sim so_q(3)$. The number operator N plays the role of the first-order invariant of $U_q(2) = SU_q(2) \otimes U(1)$. The second-order Casimir operator of the subgroup $SU_q(2)$ is

$$\begin{aligned} \boldsymbol{T}^2 &= \frac{2\Omega_j}{2}(T_+T_- + T_-T_+ + [\omega T_0][T_0+1]_\omega + [\omega T_0][T_0-1]_\omega) \\ &= 2\Omega_j(T_-T_+ + [\omega T_0][T_0+1]_\omega). \end{aligned} \tag{75}$$

Here $[X]_\omega = \frac{q^{\omega X}-q^{-\omega X}}{q^\omega - q^{-\omega}}$ and the following identity has been used:

$$[\omega T_0][T_0+1]_\omega - [\omega T_0][T_0-1]_\omega = [2\omega T_0]. \tag{76}$$

The Casimir operator coincides with the classical one in the limit $q \to 1$ (11).

(2) The other $u_q^0(2)$ subalgebra is

$$\begin{array}{ll} [K_{+1}^0, K_{-1}^0] = \left[2\dfrac{K_0^0}{2\Omega_j}\right] & [K_0^0, K_{\pm1}^0] = \pm K_{\pm1}^0 \\ [T_0, K_{\pm1}^0] = 0 & [T_0, K_0^0] = 0 \end{array} \tag{77}$$

where the generators are defined in (59).

The operator T_0 (54) commutes with the generators of $su_q^0(2)$ (77) and acts as a first-order invariant of $u_q^0(2) = su_q^0(2) \oplus u_{T_0}^0(1)$. The operators $\{K_k^0\}, k = 0, \pm 1$ couple q-deformed particles of two different kinds. The second-order Casimir operator of the subgroup $SU_q^0(2)$ is given by

$$\begin{aligned} \boldsymbol{C}_2(SU_q^0(2)) &= \frac{2\Omega_j}{2}(K_{+1}^0K_{-1}^0 + K_{-1}^0K_{+1}^0 + [\omega K_0^0][K_0^0+1]_\omega + [\omega K_0^0][K_0^0-1]_\omega) \\ &= 2\Omega_j(K_{-1}^0K_{+1}^0 + [\omega K_0^0][K_0^0+1]_\omega) \end{aligned} \tag{78}$$

which coincides with the classical invariant (15) in the limit $q \to 1$.

(3) The two mutually complementary subalgebras $su_q^+(2)$ and $su_q^-(2)$ of the algebra $sp_q(4)$ are given by the q-deformed operators (63) and the non-deformed Cartan operators (64). They have the following commutation relations:

$$\begin{array}{ll} [F_{+1}^\pm, G_{-1}^\pm] = \rho_\pm[4\omega E_0^\pm] & \\ [E_0^\pm, F_{+1}^\pm] = F_{+1}^\pm & [E_0^\pm, G_{-1}^\pm] = -G_{-1}^\pm \end{array} \tag{79}$$

with $\rho_\pm = \frac{q^{\pm1}+q^{\pm\frac{1}{2\Omega_j}}}{2}$. It is again true that each of the generators $\{F_{+1}^+, G_{-1}^+, E_0^+\}$ of $SU_q^+(2)$ commutes with all the generators of the other $SU_q^-(2)$ subgroup $\{F_{+1}^-, G_{-1}^-, E_0^-\}$. The first-order invariants $N_{\mp1}$ of $u_q^\pm(2)$ give the extension of $su_q^\pm(2)$ to the subgroup $u_q^\pm(2) = su_q^\pm(2) \oplus u^\mp(1)$. The operator $F_{+1}^\pm$ ($G_{-1}^\pm$) creates (destroys) a q-deformed pair of particles of the same kind. The Casimir invariant of the subgroup $SU_q^\pm(2)$ is

$$\begin{aligned} \boldsymbol{C}_2(SU_q^\pm(2)) &= \frac{\Omega_j}{2}(F_{+1}^\pm G_{-1}^\pm + G_{-1}^\pm F_{+1}^\pm + \rho_\pm[2\omega E_0^\pm][E_0^\pm+1]_{2\omega} \\ &\quad + \rho_\pm[2\omega E_0^\pm][E_0^\pm-1]_{2\omega}) \\ &= \Omega_j(G_{-1}^\pm F_{+1}^\pm + \rho_\pm[2\omega E_0^\pm][E_0^\pm+1]_{2\omega}). \end{aligned} \tag{80}$$

The useful identity (76) now has the form

$$[2\omega E_0^\pm][E_0^\pm+1]_{2\omega} - [2\omega E_0^\pm][E_0^\pm-1]_{2\omega} = [4\omega E_0^\pm]. \tag{81}$$

The Casimir operator coincides with the classical one (18) in the limit $q \to 1$.

The q-deformed symplectic algebra reverts back to the classical limit for the rest of the commutation relations between its generators (53),

$$\begin{aligned}
&[F_l^\dagger, G_k]_{2(k-l)} = \frac{\varphi_{l,k}}{2\sqrt{\Omega_j}} T_{l+k} q^{(l-k)\omega N_{l-k}} && l+k \neq 0\\
&[T_l, F_k^\dagger]_{k-l} = \frac{\chi_{l,k}}{\sqrt{\Omega_j}} F_{l+k}^\dagger q^{-l\omega N_{-l}} && l \neq 0\\
&[T_l, G_k]_{k-l} = -\frac{\phi_{l,k}}{\sqrt{\Omega_j}} G_{l+k} q^{l\omega N_l} && l \neq 0\\
&[T_0, F_k^\dagger] = kF_k^\dagger \qquad [T_0, G_k] = kG_k
\end{aligned} \tag{82}$$

where the constants are defined as follows:

$$\begin{aligned}
&\varphi_{\pm1,0} = 2q^{\mp2}\rho_\pm \qquad \varphi_{0,\pm1} = 2q^{\pm(2+\frac{1}{2\Omega_j})}\rho_\mp \qquad \varphi_{\pm1,\pm1} = 0\\
&\chi_{1,-1} = \rho_- \qquad \chi_{-1,1} = \rho_+ \qquad \chi_{\pm1,0} = 1 \qquad \chi_{\pm1,\pm1} = 0\\
&\phi_{1,-1} = q^{-1}\rho_- \qquad \phi_{-1,1} = q\rho_+ \qquad \phi_{\pm1,0} = q^{\mp1} \qquad \phi_{\pm1,\pm1} = 0.
\end{aligned} \tag{83}$$

Another set of the same commutation relations can be obtained, which is symmetric with respect to the exchange $q \leftrightarrow q^{-1}$:

$$\begin{aligned}
&[F_l^\dagger, G_k] = \frac{1}{2\sqrt{\Omega_j}} \frac{1}{[2]} T_{l+k} \Psi_{lk}(N_{l-k}) && l+k \neq 0\\
&[T_l, F_k^\dagger] = \frac{1}{2\sqrt{\Omega_j}} \frac{1}{[2]} F_{l+k}^\dagger \Psi_{l0}(N_k) && l, k \neq 0\\
&[T_l, G_k] = -\frac{1}{2\sqrt{\Omega_j}} \frac{1}{[2]} G_{l+k} \Psi_{0k}(N_{-k}) && l, k \neq 0\\
&[T_l, F_0^\dagger]_{\frac{[2]}{2}} = \frac{1}{2\sqrt{\Omega_j}} F_l^\dagger (q^{\omega N_{-l}} + q^{-\omega N_{-l}}) && l \neq 0\\
&[T_l, G_0]_{\frac{[2]}{2}} = -\frac{1}{2\sqrt{\Omega_j}} G_l (q^{\omega N_l} + q^{-\omega N_l}) && l \neq 0
\end{aligned} \tag{84}$$

where the functions $\Psi_{lk}(N_p)$ are defined in the following way:

$$\Psi_{lk}(N_p) = \begin{cases} q^{\omega N_p} + q^{-\omega N_p} + q^{\omega(N_p+1)-1} + q^{-\omega(N_p+1)+1} & k = 0\\ q^{\omega N_p - 1} + q^{-\omega N_p + 1} + q^{\omega(N_p-1)} + q^{-\omega(N_p-1)} & l = 0. \end{cases} \tag{85}$$

The realization of $sp_q(4)$ introduced here is consistent with the algebra of the Chevalley generators of $U_q(SO(5))$, which is given in [13]. The comparison of the commutation relations yields for the first triplet of the generators corresponding to the long root 1

$$SU_q^\tau(2) : (e_1, f_1, h_1) \leftrightarrow (T_+, T_-, \omega T_0) \tag{86}$$

and for the short root 2

$$SU_q^-(2) : (e_2, f_2, h_2) \leftrightarrow \left(\frac{F_{+1}^-}{\sqrt{[2]}}, \frac{G_{-1}^-}{\sqrt{[2]}}, \omega E_0^- \right). \tag{87}$$

The renormalization of the generators of the second triplet is introduced so that (79) can be written in the standard $su_q(2)$ form:

$$\left[\frac{F_{+1}^\pm}{\sqrt{[2]}}, \frac{G_{-1}^\pm}{\sqrt{[2]}} \right] = \rho_\pm \frac{[4\omega E_0^\pm]}{[2]} = \rho_\pm [2\omega E_0^\pm]_2. \tag{88}$$

The rest of the commutation relations of both triplets are consistent within the parameter ω. Comparing (82) with the other four generators, we obtain

$$(e_3^+, f_3^+) \leftrightarrow (F_0 q^{-\omega N_{-1}}, G_0 q^{-\omega N_{-1}}) \tag{89}$$

and

$$(e_4^+, f_4^+) \leftrightarrow (F_{+1}^+ q^{-\omega N_{-1}}, G_{-1}^+ q^{-\omega N_{-1}}) \tag{90}$$

which are determined up to an overall multiplicative constant factor. The results prove the isomorphism of the q-fermion realization of $sp_q(4)$ and all possible representations of its standard $SU_q(2)$ subgroup to the triplets of the Chevalley generators associated with the shorter and longer roots of $U_q(SO(5))$.

3.3. *Action space of the fermion realization of $sp_q(4)$*

In general, the q-deformed fermion operators (70) act as in the classical case in a finite metric space $\mathcal{E}_j$ for each particular j-level, with a vacuum $|0\rangle$ defined by $\alpha_{m,\sigma}\ |0\rangle = 0$. The scalar product in $\mathcal{E}_j$ is chosen in such a way that $\alpha_{m,\sigma}^\dagger$ is a Hermitian conjugate to $\alpha_{m,\sigma} : (\alpha_{m,\sigma}^\dagger)^* = \alpha_{m,\sigma}$, and $\langle 0|0\rangle = 1$. In general the q-deformed states are different from the classical ones, but reduce to the classical ones in the limit $q \to 1$.

The q-deformed creation (annihilation) operators $F_k^\dagger, k = 0, \pm 1$ (53) are components of a tensor of rank 1 with respect to the subgroup $SU_q^T(2)$ (84). These operators create a pair of q-fermions coupled to a total angular momentum $J = 0$ and a total isospin $T = 1$. Analogous to the classical limit, a set of vectors that span each space $\mathcal{E}_j^+$ in the q-deformed case can be chosen to be of the form

$$|n_1, n_0, n_{-1})_q = (F_1^\dagger)^{n_1}(F_0^\dagger)^{n_0}(F_{-1}^\dagger)^{n_{-1}}|0\rangle. \tag{91}$$

The basis is obtained by orthonormalization of (91). The index q will be dropped from the notation for the basis states in the following cases, which treat only the deformed space.

As in the classical case, Ω_j labels the representation for each particular j-shell. The basis states are uniquely specified by the classification schemes which use the $su_q(2)$ subalgebras and the relevant Cartan generators. In the q-deformed case the Cartan generators of $Sp_q(4)$ can be chosen to be the nondeformed operators $N_{\pm 1}$ or their equivalent set of operators N and $T_0 \equiv \tau_0$ (54). The eigenvalues of these operators that label the basis states coincide with the ones in the classical case and the example of table 1 can still be used. The quantum numbers provided by the eigenvalues of the q-deformed Casimir invariants have to be taken in the limit $q \to 1$.

We briefly list the reduction chains and compare them with their classical counterparts in order to emphasize the similarity and differences between them. The basis states together with the second-order Casimir operators and their eigenvalues are often used in physical applications. It is in this sense that their q-deformation may lead to some interesting new results.

(1) In the limit $q \to 1$, the second-order Casimir operator, $\boldsymbol{T}^2$, of the $SU_q^T(2)$ subgroup has the eigenvalues

$$\boldsymbol{T}^2|n, T, i\rangle_q \underset{q\to 1}{\to} T(T+1)|n, T, i\rangle \tag{92}$$

where $T = \frac{\tilde{n}}{2}, \frac{\tilde{n}}{2} - 2, \ldots, 1$ (odd) or 0 (even), where $\tilde{n} = \min\left\{n, 4\Omega_j - n\right\}$ and $i = -T, -T+1, \ldots, T$. In the deformed case the eigenvalues of $\boldsymbol{T}^2$ for the lowest- and the highest-weight states (75) are

$$\boldsymbol{T}^2|n, T, \pm T\rangle_q = 2\Omega_j \left[\frac{1}{2\Omega_j}\right][T]_\omega[T+1]_\omega|n, T, \pm T\rangle_q. \tag{93}$$

(2) The reduction chain $Sp_q(4) \supset SU_q^0(2) \otimes U_q(1)_{T_0}$ describes pairing between fermions of different types and introduces the seniority quantum number $2(n_1 + n_{-1})_{\max}$ in the labelling scheme for the basis states, $|i, 2(n_1 + n_{-1})_{\max}, n)$. The eigenvalue of the second-order Casimir operator for this q-deformed subalgebra is given by

$$C_2(SU_q^0(2))|n_1, n_0, n_{-1}) = 2\Omega_j \left[\frac{1}{2\Omega_j}\right] \left[\frac{2\Omega_j - 2(n_1 + n_{-1})_{\max}}{2}\right]_\omega \times \left[\left(\frac{2\Omega_j - 2(n_1 + n_{-1})_{\max}}{2} + 1\right)\right]_\omega |n_1, n_0, n_{-1}). \quad (94)$$

Here again, the generators of the subalgebra $su_q^0(2)$ act along the columns:

$$\begin{aligned} K_{+1}^0|n_1, n_0, n_{-1}) &= |n_1, n_0 + 1, n_{-1}) \\ K_{-1}^0|n_1, n_0, n_{-1}) &= [n_0]_{\frac{1}{2\Omega_j}} \left[1 - \frac{2(n_{-1} + n_1) + n_0 - 1}{2\Omega_j}\right] |n_1, n_0 - 1, n_{-1}) \\ N|n_1, n_0, n_{-1}) &= 2(n_{-1} + n_1 + n_0)|n_1, n_0, n_{-1}). \end{aligned} \quad (95)$$

The normalized basis states,

$$|n_1, n_0, n_{-1}\rangle = \frac{1}{\mathcal{M}_0(n_1, n_0, n_{-1})} |n_1, n_0, n_{-1}) \quad (96)$$

can be derived from (95). For the three types of state in this reduction, the normalization coefficients are

$$\begin{aligned} \mathcal{Q}_0^2(n_1, n_0, n_{-1}) &= [n_0]_\omega! \prod_{k=0}^{n_0-1} \left[1 - \frac{2(n_{-1} + n_1) + k}{2\Omega_j}\right] \\ \mathcal{M}_0(0, n_0, 0) &= \mathcal{Q}_0(0, n_0, 0) \\ \mathcal{M}_0(n_1, n_0, 0) &= \mathcal{Q}_0(n_1, n_0, 0)\mathcal{M}(n_1) \\ \mathcal{M}_0(0, n_0, n_{-1}) &= \mathcal{Q}_0(0, n_0, n_{-1})\mathcal{M}(n_{-1}) \end{aligned} \quad (97)$$

where the q-deformed factorial is defined by $[A]_k! = [A]_k[A - 1]_k \ldots 1$. The normalization coefficients $\mathcal{M}(n_{\pm 1})$ of the lowest-weight state ($n_0 = 0$ and $n_{\mp 1} = 0$) in each representation are derived by means of the generators of the next reduction.

(3) The other reduction $Sp_q(4) \supset U_q(2)_{N_\mp} \supset SU_q^\pm(2) \supset U_q(1)_{N_\pm}$ introduces deformation in the model of coupled fermions of the same kind. Here the labelling is $|n_{\mp 1}, n_0 = n_{0\max}, n_{\pm 1})$, where $n_{0\max} = \{0 \text{ or } 1\}$ is the seniority quantum number. The action of the Casimir operator on the states is given by

$$C_2(SU_q^\pm(2))|n_1, n_0, n_{-1}) = \rho_\pm\, \Omega_j \left[\frac{1}{\Omega_j}\right] \left[\frac{\Omega_j - n_{0\max}}{2}\right]_{2\omega} \times \left[\frac{\Omega_j - n_{0\max}}{2} + 1\right]_{2\omega} |n_1, n_0, n_{-1}). \quad (98)$$

In the deformed case the action of the Casimir invariant of $SU_q^+(2)$ differs from that of the Casimir invariant of $SU_q^-(2)$ by the factor ρ_+/ρ_-. The generators of $su_q^\pm(2)$ transform the states along the diagonals as

$$\begin{aligned} F_{+1}^\pm|n_1, n_0, n_{-1}) &= |n_{\pm 1} + 1, n_0, n_{\mp 1}) \\ G_{-1}^\pm|n_1, n_0, n_{-1}) &= \rho_\pm[n_{\pm 1}]_{\frac{1}{\Omega_j}} \left[1 - \frac{n_{\pm 1} + n_0 - 1}{\Omega_j}\right] |n_{\pm_1} - 1, n_0, n_{\mp 1}) \\ N_\pm|n_1, n_0, n_{-1}) &= (2n_{\pm 1} + n_0)|n_1, n_0, n_{-1}). \end{aligned} \quad (99)$$

The normalized basis states,

$$|n_1, n_0, n_{-1}\rangle = \frac{1}{\mathcal{M}_\pm(n_1, n_0, n_{-1})}|n_1, n_0, n_{-1}) \tag{100}$$

can be derived from (99). For the two types of state $n_0 = \{0 \text{ or } 1\}$ in this reduction, the normalization coefficients are

$$\mathcal{M}^2_\pm(n_1, n_0, n_{-1}) = \rho_+\rho_- [n_1]_{2\omega}![n_{-1}]_{2\omega}! \prod_{l=0}^{n_1-1}\left[1 - \frac{n_0 + l}{\Omega_j}\right] \prod_{l=0}^{n_{-1}-1}\left[1 - \frac{n_0 + l}{\Omega_j}\right] \tag{101}$$

where for $n_0 = 0$ and $n_{\mp 1} = 0$ it follows that

$$\mathcal{M}^2(n_{\pm 1}) = \rho_\pm [n_{\pm 1}]_{2\omega}! \prod_{l=0}^{n_{\pm 1}-1}\left(1 - \frac{l}{\Omega_j}\right). \tag{102}$$

It is important to emphasize that this deformation may lead to basis states whose content is very different from the classical case since there is no known simple function that transforms the classical fermion operators $c^\dagger_{m,\sigma}$ and $c_{m,\sigma}$ into the q-deformed ones $\alpha^\dagger_{m,\sigma}$ and $\alpha_{m,\sigma}$. A smooth function may not exist when the anticommutation relations (73) hold simultaneously with both signs for one and the same σ, as they are defined in (70).

The deformed basis states are labelled by the classical eigenvalues of the invariant operators of the reduction along each of the cases considered. The matrix elements, particularly of the raising and lowering generators of $sp_q(4)$ and the second-order invariants, are also deformed, which leads to different results in physical applications. After obtaining the correspondence between the q-fermion realization of $sp_q(4)$ and the Chevalley generators of $U_q(SO(5))$ we can compare the two bases for an irreducible representation $\Omega_j = \frac{1}{4}n_{\max}$, which corresponds to the representation (n_1, n_2) at $n_1 = n_2 = \frac{1}{4}n_{\max}$ [13]. In the classical and in the q-deformed cases, the first basis considered in [13] is related to the basis states (24) and (91).

4. Conclusion

In this paper we consider a fermion realization of the $sp(4)$ algebra and its deformations. The original algebra, as well as some of its deformed realizations, act in the same finite space $\mathcal{E}_j$. The finiteness of the representations is due to the Pauli principle.

The deformed realization of $sp_q(4)$ is based on the standard q-deformation of the two-component Clifford algebra [14], realized in terms of creation and annihilation fermion operators. For the $sp(4)$ case eight of the ten generators are deformed, the fermion number operators N_1 and N_{-1} and their linear combinations being the exceptions. The deformed generators of $Sp_q(4)$ close different realizations of the compact $u_q(2)$ subalgebra. The induced representations of each $u_q(2)$ are reducible in the space $\mathcal{E}^+_j$ and decompose into irreducible representations. In this way we obtain a full description of the IURs of $U_q(2)$ of four different realizations of $u_q(2)$: $u^\tau_q(2)$, $u^0_q(2)$ and $u^\pm_q(2)$.

Each reduction into compact subalgebras of $sp(4)$ and its deformations affords the possibility of a description of a different physical model with different dynamical symmetries. While within a particular deformation scheme the basis states may either be deformed or not, the generators are always deformed as is their action on basis states. With a view towards applications, the additional parameter of the deformation gives a richer variety of operators associated with observables, nondeformed as well as deformed. In a Hamiltonian theory this implies a dependence of the matrix elements on the deformation parameter, leading to the possibility of greater flexibility and richer structures within the framework of algebraic descriptions.

Acknowledgments

This work was supported in part by the US National Science Foundation under grant PHY-9970769 and Cooperative Agreement EPS-9720652 that includes matching from the Louisiana Board of Regents Support Fund.

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